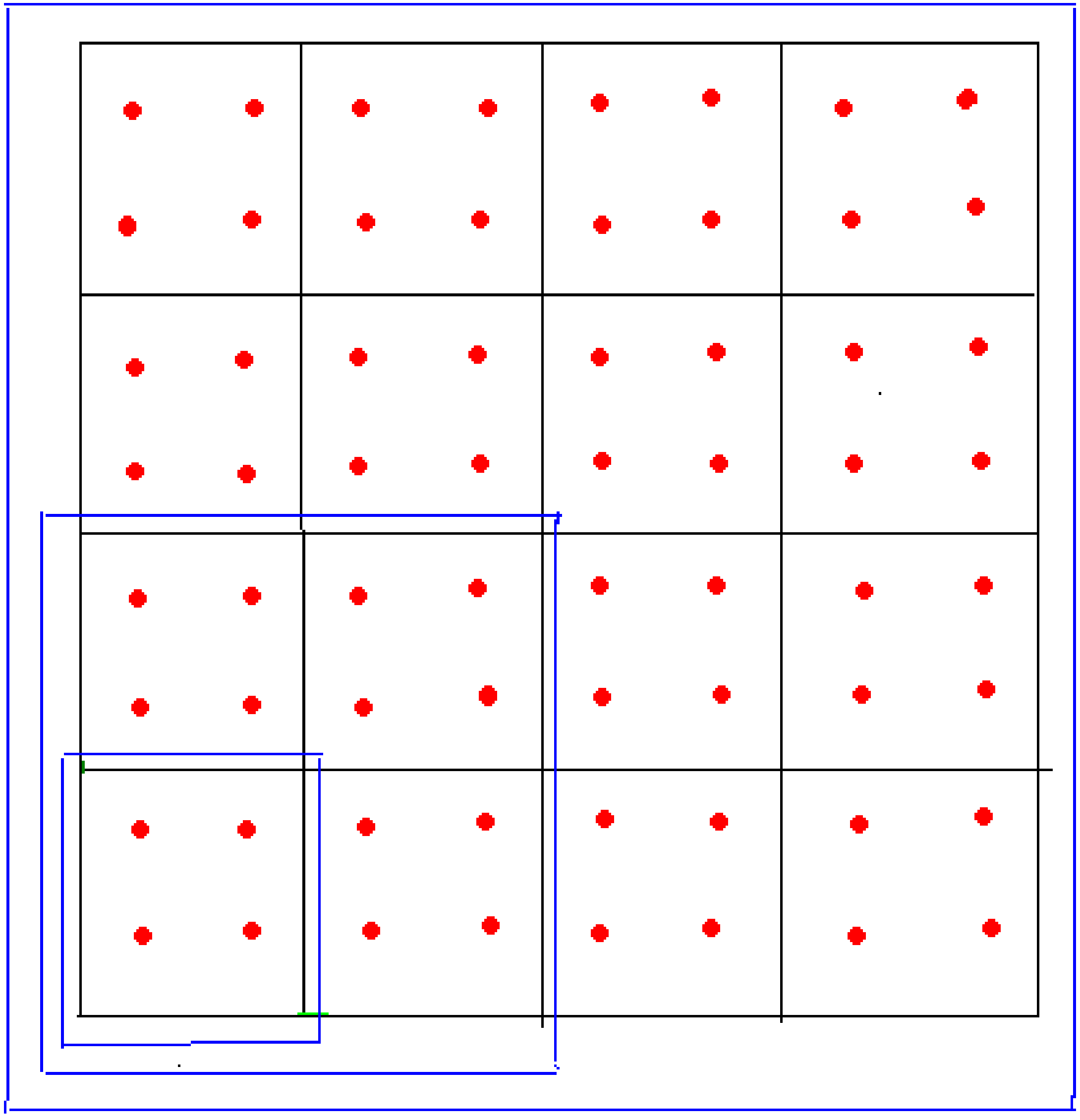
Exactly solvable renormalization group model and new conjectures in general renormalization group theory

M.D. Missarov, Kazan Federal University

Let Z^d denote the lattice of integer-valued vectors of R^d, $k = (k_1, \ldots, k_d) \in Z^d$, $s \in \mathbf{N}$, $p \in \mathbf{N}$, $V_{k,s} = \{j \in Z^d : (k_l - 1)p^s < j_l \le k_l p^s,\ l = 1, \ldots, d\}$. The hierarchical distance $d_p(i,j)$, $i, j \in Z^d$ over Z^d is defined as $d_p(i,j) = p^{s(i,j)}$, if $i \ne j$; $s(i,j) = \min\{s :$ there is k such that $i \in V_{k,s},\ j \in V_{k,s}\}$. The lattice Z^d endowed with hierarchical distance $d_p(i,j)$ will be called the d-dimensional hierarchical lattice Λ.



The fermionic hierarchical model on the hierarchical lattice Λ is specified by the Hamiltonian

$$H(\psi^*;\alpha) = \sum_{i,j\in\Lambda} d_p^{-\alpha}(i,j)[\bar{\psi}_1(i)\psi_1(j) + \bar{\psi}_2(i)\psi_2(j)] +$$

$$\sum_{i\in\Lambda} L(\psi^*;r,g),$$

where

$$L(\psi^*(i);r,g) = r\left(\bar{\psi}_1(i)\psi_1(i) + \bar{\psi}_2(i)\psi_2(i)\right) +$$

$$g\bar{\psi}_1(i)\psi_1(i)\bar{\psi}_2(i)\psi_2(i).$$

Four-component spins $\psi^*(i) = (\bar{\psi}_1(i), \psi_1(i), \bar{\psi}_2(i), \psi_2(i))$, whose components are the generators of the Grassmann algebra, are located at the nodes of this lattice.

It is convenient to use the concept of the Grassman-valued "density" of the free measure

$$f(\psi^*) = \exp\{-L(\psi^*; r, g)\}$$

instead of the lagrangian $L(\psi^*; r, g)$.

In the general case, the "density" of the free measure is given by

$$f(\psi^*;c) = c_0 + c_1(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2) + c_2\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2,$$

$c = (c_0, c_1, c_2) \in R^3$. If $c_0 \neq 0$ (the regular case), then the coupling constants r and g are related to c by the formulas

$$r(c) = -\frac{c_1}{c_0}, \quad g(c) = \frac{c_1^2 - c_0c_2}{c_0^2}.$$

If $c_0 = 0$ (for instance, as in the case where the density is given by the Grassmann δ-function $\delta(\psi^*) = \bar{\psi}_1\psi_1\bar{\psi}_2\psi_2$), then the exponential representation is impossible.

The triple (c_0, c_1, c_2) can be naturally treated as a point in the two-dimensional real projective space RP^2 because two triples that differ by a nonzero factor represent the same Gibbs state.

The block-spin transformation of the Kadanoff-Wilson renormalization group (RG) is defined by the formula

$$r(\alpha)\psi^*(i) = p^{-\alpha/2} \sum_{j\in V_{i,1}} \psi^*(j),$$

where α is renormalization group parameter.

The Gaussian part of the model Hamiltonian is invariant under this RG transformation RG.

In the non-Gaussian it reduces to the to the transformation $R(\alpha)$ of the coupling constants $R(\alpha)(r, g) = (r', g')$,

$$r' = \lambda\left(\frac{(r+1)^2 - g}{(r+1)^2 - g/n}(r+1) - 1\right),$$

$$g' = \frac{\lambda^2}{n}\left(\frac{(r+1)^2 - g}{(r+1)^2 - g/n}\right)^2 g,$$

where $\lambda = p^{\alpha - d}$, $n = p^d$ is the size of the elementary cell of the hierarchical lattice Λ.

The RG transformation in the space of the free measure "density" is also denoted by $R(\alpha)$: $R(\alpha)(c_0, c_1, c_2) = (c'_0, c'_1, c'_2)$,

$$c'_0 = \left((c_1 - c_0)^2 + \frac{1}{n}(c_0 c_2 - c_1^2)\right),$$

$$c'_1 = \lambda\left((c_1 - c_0)(c_2 - c_1) + \frac{1}{n}(c_0 c_2 - c_1^2)\right),$$

$$c'_2 = \lambda^2\left((c_2 - c_1)^2 + \frac{1}{n}(c_0 c_2 - c_1^2)\right).$$

The RG transformation in the c-space seems more aesthetic and allows visualizing the picture of the dynamics in the entire space because the projective space is compact and allows eliminating some singularities in (r, g) coordinates. Indeed, the first map is not defined at the points of the critical parabola $g = n(r+1)^2$, but formulas of the RG transformation in the c space allow defining it. Points of the critical parabola under the first iteration of RG map go to the non-regular domain, but under the second iteration they return to the regular domain.

The transformation $R(\alpha)$ has the trivial (Gaussian) fixed point $(0, 0)$ and two non-Gaussian fixed points in the (r, g) coordinates. In the c coordinates, the Gaussian point can be written as $(1, 0, 0)$, and one more fixed point $(0, 0, 1)$ can be seen, given the Grassmann δ function $\delta(\psi^*) = \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2$. We denote this fixed point δ.

The RG transformation has two regular non-Gaussian fixed points that for $\alpha \neq d$ are given in the (r, g) coordinates by the formulas:

$$r_+(\alpha) = \frac{p^{d/2} - p^{\alpha-d}}{1 - p^{d/2}}, \quad g_+(\alpha) = \frac{r_+(\alpha)(1 + r_+(\alpha))^2}{1 + r_+(\alpha) + p^{-d/2}}, \ ,$$

$$r_-(\alpha) = \frac{-p^{d/2} - p^{\alpha-d}}{1 + p^{d/2}}, \quad g_-(\alpha) = \frac{r_-(\alpha)(1 + r_-(\alpha))^2}{1 + r_-(\alpha) - p^{-d/2}}.$$

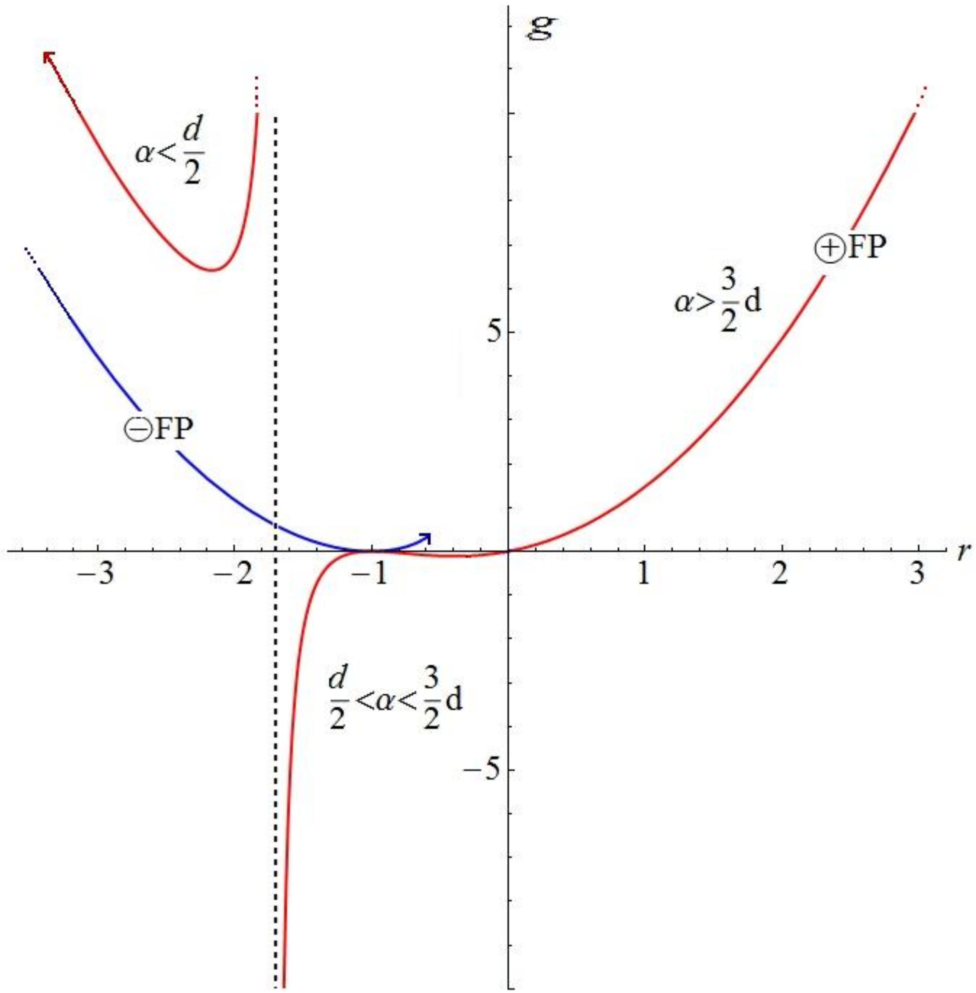


Fig. 1

Trajectories of "+ " and "- " fixed points in (r,g)- plane

The map $R(\alpha)$ itself, like the map from RP^2 to RP^2, is well defined everywhere except the point $(1, 1, 1)$ because $R(\alpha)(1, 1, 1) = (0, 0, 0)$. It is given by the coordinates $(-1, 0)$ in the (r, g) plane. We call this point the singular point of the RG transformation.

One can see, that Grassmann Fourier transform transposes the coefficients c_0, c_1, c_2 of the density $f(\eta^*; c_0, c_1, c_2)$:

$$F_{\eta^* \to \xi^*}(f(\eta^*; c_0, c_1, c_2)) =$$

$$= \int \exp\{-(\bar{\xi}_1\eta_1 + \bar{\xi}_2\eta_2 + \xi_1\bar{\eta}_1 + \xi_2\bar{\eta}_2)\}$$

$$f(\eta^*; c_0, c_1, c_2)\, d\eta_1\, d\bar{\eta}_1\, d\eta_2\, d\bar{\eta}_2 =$$

$$= f(\xi^*; c_2, c_1, c_0).$$

It is easy to verify commutation on relation

$$FR(\alpha) = R(2d - \alpha)F.$$

Therefore it is sufficient to investigate the case $\alpha > d$.

Hereafter, we assume that $\alpha > d$.

We consider the realization of the projective c space in the form of the hemisphere

$$S = \left\{(c_0, c_1, c_2) : c_0^2 + c_1^2 + c_2^2 = 1, c_0 \geq 0\right\},$$

where the opposite points of the boundary circle $c_1^2+c_2^2 = 1$ are identified. To obtain the flat (two-dimensional) picture, we use the orthogonal projection S on the disk $D = \left\{(c_1, c_2) : c_1^2 + c_2^2 \leq 1\right\}$.

The regular point (r, g) then corresponds to $(c_1(r, g), c_2(r, g))$,

$$c_1(r, g) = -\frac{r}{\sqrt{1 + r^2 + (r^2 - g)^2}},$$
$$c_2(r, g) = \frac{r^2 - g}{\sqrt{1 + r^2 + (r^2 - g)^2}}.$$

We note that the points $(c_1(r, g), c_2(r, g))$ belong to the interior of the disk D. The trivial fixed point $r = 0$, $g = 0$ is also represented by the point $(0, 0)$ in the (c_1, c_2) coordinates. The fixed point δ in (c_1, c_2) coordinates is determined by the point $(0, 1)$.

The line $g = 0$ in the (c_1, c_2) space is described by the curve $l_0 = \{(c_1(r,0), c_2(r,0))\,; r \in R\}$. Completing the curve l_0 with the limit point $(0,1)$, we obtain a closed curve l. The lower half plane $\{(r,g) : g < 0\}$ is given in the (c_1, c_2) space by the region D_1 bounded by the curve l. The upper half-plane $\{(r,g) : g > 0\}$ is mapped to the interior part of the domain $D \setminus D_1$.

The Fig.3 demonstrates the disk D and the attraction domains of the fixed point $\delta = (0, 1)$ when $\alpha = 1, 7$, $p = 2$, $d = 1$. Almost all points of $D \setminus D_1$ are attracted to the fixed point δ. The points are colored red if they tend to δ from the left. The points are colored blue if they tend to δ from the right. The singular point is marked by the cross. In the red zone we see a large domain $A(0)$ and countable series of nonintersecting subsets $A(1)$, $A(2)$, $A(3)$, (satellite domains) .

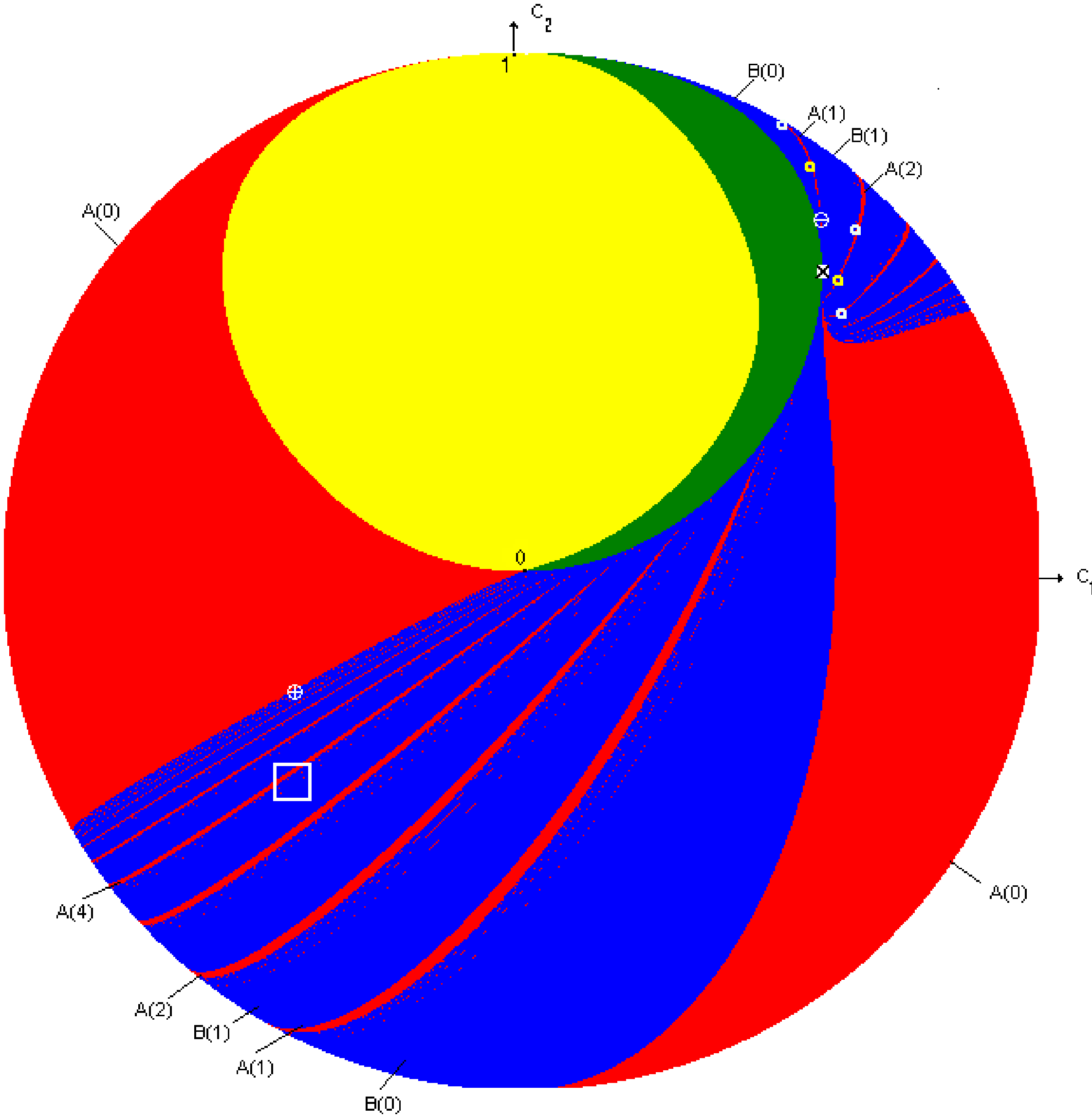


Fig.3

Signes "+", "-", 0, 1 indicate the locations of "+", "-", trivial Gaussian fixed points and fixed point at infinity correspondingly in projective coordinates, "x" indicates the location of singular point, other five points are points of the 2-nd and 3-rd order cycles. Yellow and green parts are RG-invariant sets of the domain which corresponds to the lower half-plane in (r,g)-coordinates.

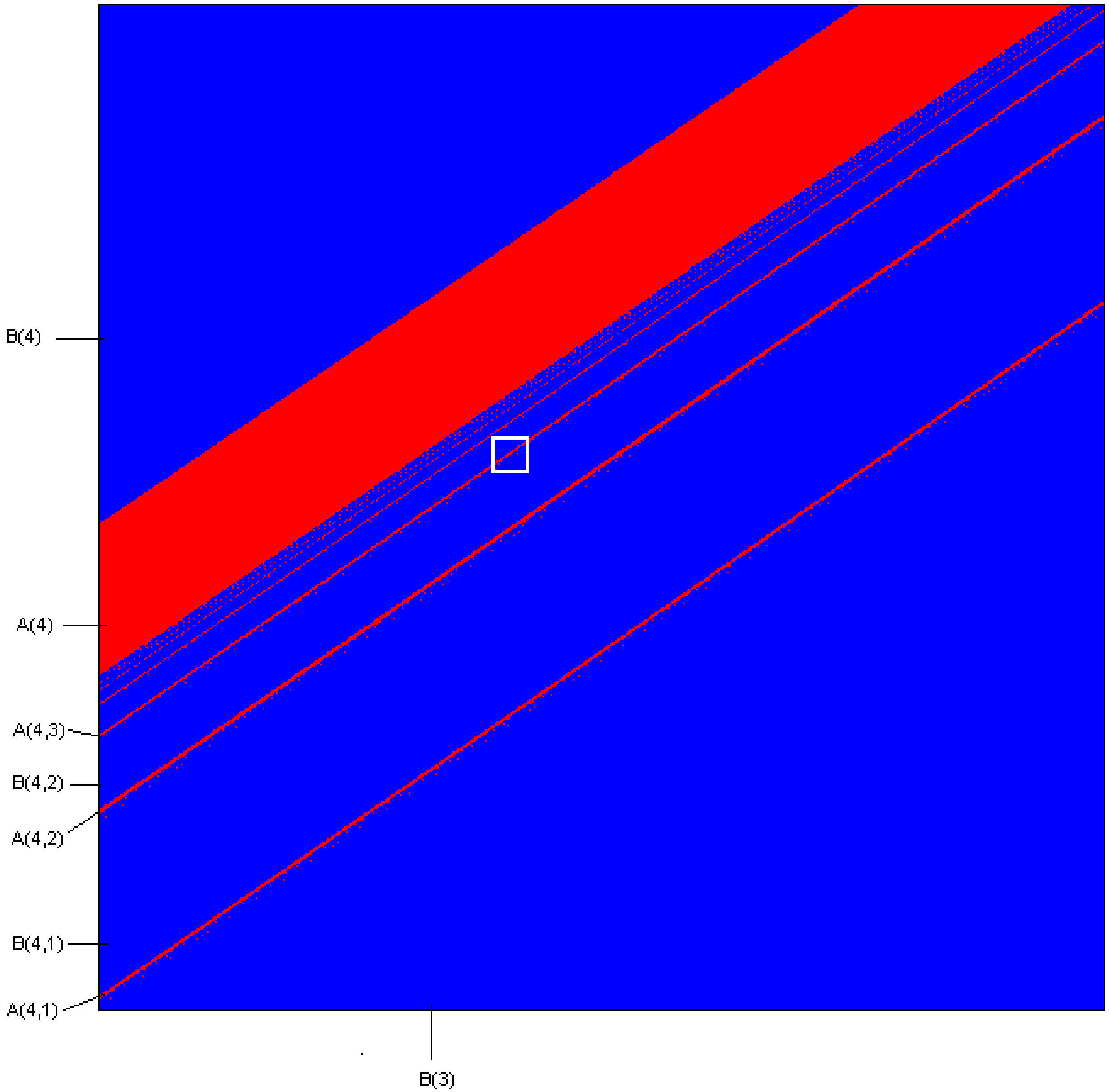


Fig. 4

Enlarged picture of the square, selected on the Fig.3

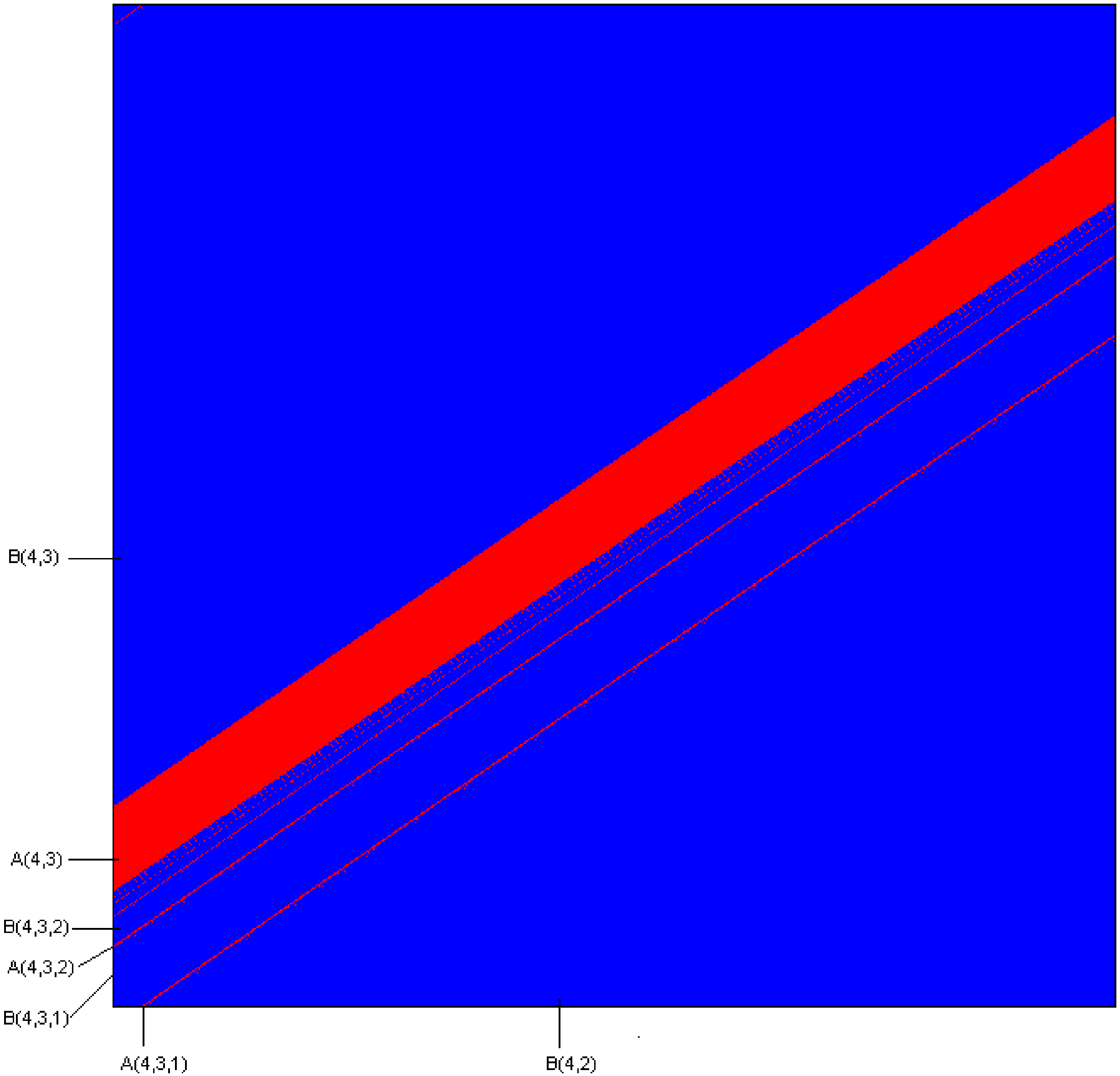


Fig.5

Enlarged picture of the square, selected on the Fig.4

In turn, every of domains $A(i), i = 1, 2,$ has its own (satellite) countable series of nonintersecting subsets $A(i, j), j = 1, 2,$ and so on. If point belongs to zone $A(i_1, i_2, , i_k)$ and $i_1 > 1$, then after one RG-iteration it goes to zone $A(i_1 - 1, i_2, , i_k)$. If $i_1 = 1$ then it goes to $A(i_2, i_3, , i_k)$. Point from zone $A(i), i = 1, 2,$ after one RG-iteration goes to $A(i - 1)$.

$$A(i_1, i_2, , i_k) \to A(i_1 - 1, i_2, , i_k) \to ...$$

$$\to A(1, i_2, , i_k) \to A(i_2, , i_k) \to ...$$

$$\to A(i_3, i_4, , i_k) \to ... \to A(0).$$

Structure of the blue subsets $B(i_1, i_2, , i_k)$ of the points attracted to the δ from the right is analogues to the red subsets. All other points of $D \setminus D_1$ lie on the boundaries of zones $A(i_1, i_2, , i_k)$ and $B(i_1, i_2, , i_k)$. These boundaries are invariant curves of the map $R(\alpha)$ or its degrees. Numerically it is found that there are cycles of RG-map of the order k for $k < 10$ and they lie on the boundaries of sets $A(k, k, k, ...)$ and $B(k, k, k, ...)$.

If p is some prime number, it is natural to define d-dimensional hierarchical lattice as a lattice T_p^d of p-adic fractional d-dimensional vectors. Every p-adic number $x \in Q_p$ can be represented in the form

$$x = c_{-l}p^{-l} + \ldots + c_{-1}p^{-1} + c_0 + c_1 p + \ldots ,$$

where the coefficients c_i are integer numbers between 0 and $(p-1)$, and $c_{-l} \neq 0$ for some integer l.

The norm $|x|_p = p^l$ and the fractional part of x is defined as

$$\{x\}_p = c_{-l}p^{-l} + \ldots + c_{-1}p^{-1}.$$

For $x = (x_1, \ldots, x_d) \in Q_p^d$, we set $|x|_p = \max_i |x_i|_p$, $\{x\}_p = (\{x_1\}_p, \ldots, \{x_d\}_p)$. Then the discrete set $T_p^d = \{x \in Q_p^d : x = \{x\}_p\}$ can be viewed as a hierarchical lattice with the size of elementary cell $n = p^d$ and with the hierarchical distance $d_p(i,j) = |i-j|_p$, $i, j \in T_p^d$.

The continuous version of the fermionic hierarchical model is described in terms of the four–component field $\psi^*(x) = (\overline{\psi}_1(x), \psi_1(x), \overline{\psi}_2(x), \psi_2(x))$ over Q_p^d, whose components are generators of the Grassmann algebra. Let the Gibbs state describing this field be specified by the Hamiltonian

$$H(\psi^*;\alpha;r,g) = H_0(\psi^*;\alpha) + \int L(\psi^*(x);r,g)\,dx,$$

$$H_0(\psi^*;\alpha) =$$

$$c(\alpha)\int ||x-y||_p^{-\alpha}\left(\overline{\psi}_1(x)\psi_1(y) + \overline{\psi}_2(x)\psi_2(y)\right)\,dxdy,$$

where dx is the Haar measure on Q_p^d, $c(\alpha)$ is some normalizing factor. The Lagrangian is

$$L(\psi^*(x);u,v) = u(\overline{\psi}_1(x)\psi_1(x) + \overline{\psi}_2(x)\psi_2(x)) +$$

$$+v(\overline{\psi}_1(x)\psi_1(x)\overline{\psi}_2(x)\psi_2(x)).$$

The Gaussian Hamiltonian is invariant w.r.t.the group of scaling transformations $(S_\mu(\alpha)\psi^*)(x) = |\mu|_p^{d-\alpha}\psi^*(\mu x)$, where $\alpha \in R$ is the parameter of this group, μ is any p-adic number.

The value $\alpha = d+2$ corresponds to the Laplace operator in the Euclidean case or its p-adic analog.

In the space of the Hamiltonians the transformation $S_\mu(\alpha)$ acts by scaling the coupling constants:

$$S_\mu(\alpha)(u, v) = (|\mu|_p^{\alpha-d}u, |\mu|_p^{2\alpha-3d}v).$$

The discretization of the field ψ^* is the field ξ^* over T_p^d such that

$$\xi^*(j) = \int \psi^*(j + x)\chi(x)\, dx, \quad j \in T_p^d.$$

Here T_p^d is a lattice of p-adic fractional vectors (hierarchical lattice), $\chi(x)$ - characteristic function of the ball $Z_p^d = \{x \in Q_p^d : ||x||_p \leq 1\}$.

The discretization ζ^* of the field $(S_{p^{-1}}(\alpha)\psi^*)(x)$ is hierarchical block-spin renormalization group transformation over the field ξ^*,

$$\zeta^*(j) = (r(\alpha)\xi^*)(j) = p^{-\alpha/2} \sum_{i \in B(j)} \xi^*(i),$$

where $B(j) = \{i \in T_p^d : ||i - p^{-1}j||_p \leq p\}$ are the elementary blocks in the hierarchical lattice.

Discretization of the Gaussian field fermionic field ψ^* with the Hamiltonian

$$H(\psi^*;\alpha;u,v) = H_0(\psi^*;\alpha) + \int L(\psi^*(x);u,v)\,dx,$$

is a discrete fermionic field $\xi^*(j)$ with the Hamiltonian

$$H'(\xi^*;\alpha;r,g) = H_0'(\xi^*;\alpha) + \sum_{j\in T_p^d} L(\xi^*(j);r,g),$$

with the Gaussian part

$$H_0'(\xi^*;\alpha) = \sum_{i,j} h(i,j;\alpha)\,\left(\overline{\xi}_1(i)\xi_1(j) + \overline{\xi}_2(i)\xi_2(j)\right),$$

$$h(i,j;\alpha) = c_1(\alpha)\,(1-\delta_{i,j})\,||i-j||_p^{-\alpha} + c_1(\alpha)\delta_{i,j}.$$

Thus, the discretization of the continuum field leads to the hierarchical model with the same potential $L(\xi^*; r, g)$, where the coupling constants $r = r(u, v)$ and $g = g(u, v)$ of the lattice field depends on the coupling constants u and v of the continuum model and are given by non-Gaussian functional integral.

Denote the discretization transformation

$$(u, v) \to (r(u, v), g(u, v))$$

by $P(\alpha)$.

As we know, the transformation of the hierarchical RG $r(\alpha)$ can be computed explicitly in the space of coupling constants of the hierarchical model and is given by the mapping

$$R(\alpha)(r,g) = (r',g')$$
$$r' = p^{\alpha-d}\left(\frac{(r+1)^2-g}{(r+1)^2-g/p^d}(r+1)-1\right),$$
$$g' = p^{2\alpha-3d}\left(\frac{(r+1)^2-g}{(r+1)^2-g/p^d}\right)^2 g,$$

Taking into account that $r(\alpha)$ is the discrete version of scaling transformation $S_{p^{-1}}(\alpha)$, we have

$$R(\alpha)P(\alpha) = P(\alpha)S(\alpha),$$

where

$$S(\alpha)(u, v) = (p^{\alpha-d}u, p^{2\alpha-3d}v).$$

The mapping $S(\alpha)$ is given by the diagonal matrix whose eigenvalues are the eigenvalues of the differential of $R(\alpha)$ at the origin. Hence we can treat the mapping $P(\alpha)$ as a normalizing transformation to the mapping $R(\alpha)$ at zero point an can find functional integral $P(\alpha)$ as a solution of the classical functional equation.

For $\alpha > 3d/2$ both eigenvalues are more than 1, and we are in the domain where the classical Poincare theorem applies. According to to this theorem, the mapping $P(\alpha)$ can be expanded in a power series in u and v that converges for sufficiently small u and v provided α is a non-resonance value.

The resonance values in the domain $3d/2 < \alpha < 2d$ are arranged in the discrete series

$$\alpha_k = (\frac{3}{2} + \frac{1}{2(2k-1)})d, \quad k = 1, 2, \ldots.$$

and exactly corresponds to the ultraviolet poles of p-adic Feynman amplitudes. If $d \leq \alpha \leq 3d/2$, the second eigenvalue is less then 1 and we are in the so-called Siegel domain. In that case any rational α is a resonance value and the convergence of the mapping $P(\alpha)$ requires some Diophantine condition.

The renormalization procedure can be defined as the mapping inverse to the normal mapping $P(\alpha)$. We can restore coupling constants of the continuum theory from the coupling constants of the discrete model using inverse map $P^{-1}(\alpha)$.

We can prove rigorously that discrete model is well defined for the whole plane of coupling constants and almost all values of α. But we can prove rigorously that continuum model is well defined only in some small neighborhood of trivial (zero) fixed point of renormalization group. In other words the continuum model is related with the discrete model as the normal form is related with the map.

We describe several new interesting phenomena discovered in our model. Some of them can be generalized for the other bosonic and fermionic models.

1. Interpretation of renormalization procedure as a normal form to the renormalization group transformation at zero fixed point. This interpretation is valid for the bosonic hierarchical model .

2. New branch of fixed points and cycles of renormalization group. It is possible to construct locally the non-Gaussian branch of fixed points which bifurcates from the Gaussian for models in the bosonic hierarchical and Euclidean. Will be very interesting to find another branch of fixed points in the fermionic Euclidean model.

3.Commutative relation between renormalization group and Fourier transformations $F\,R(\alpha) = R(2d-\alpha)F$. This relation is true for p-adic and Euclidean case.

4. "+"-branch of fixed points lies in the lower half-plane for $d < \alpha \leq 3d/2$. As it follows from the property 3 the non-Gaussian branch of fixed points in the bosonic case is well defined for $\alpha \leq d/2$ and bifurcates from the fixed point at infinity which corresponds to constant (zero)random field. But what about $d < \alpha \leq 3d/2$?

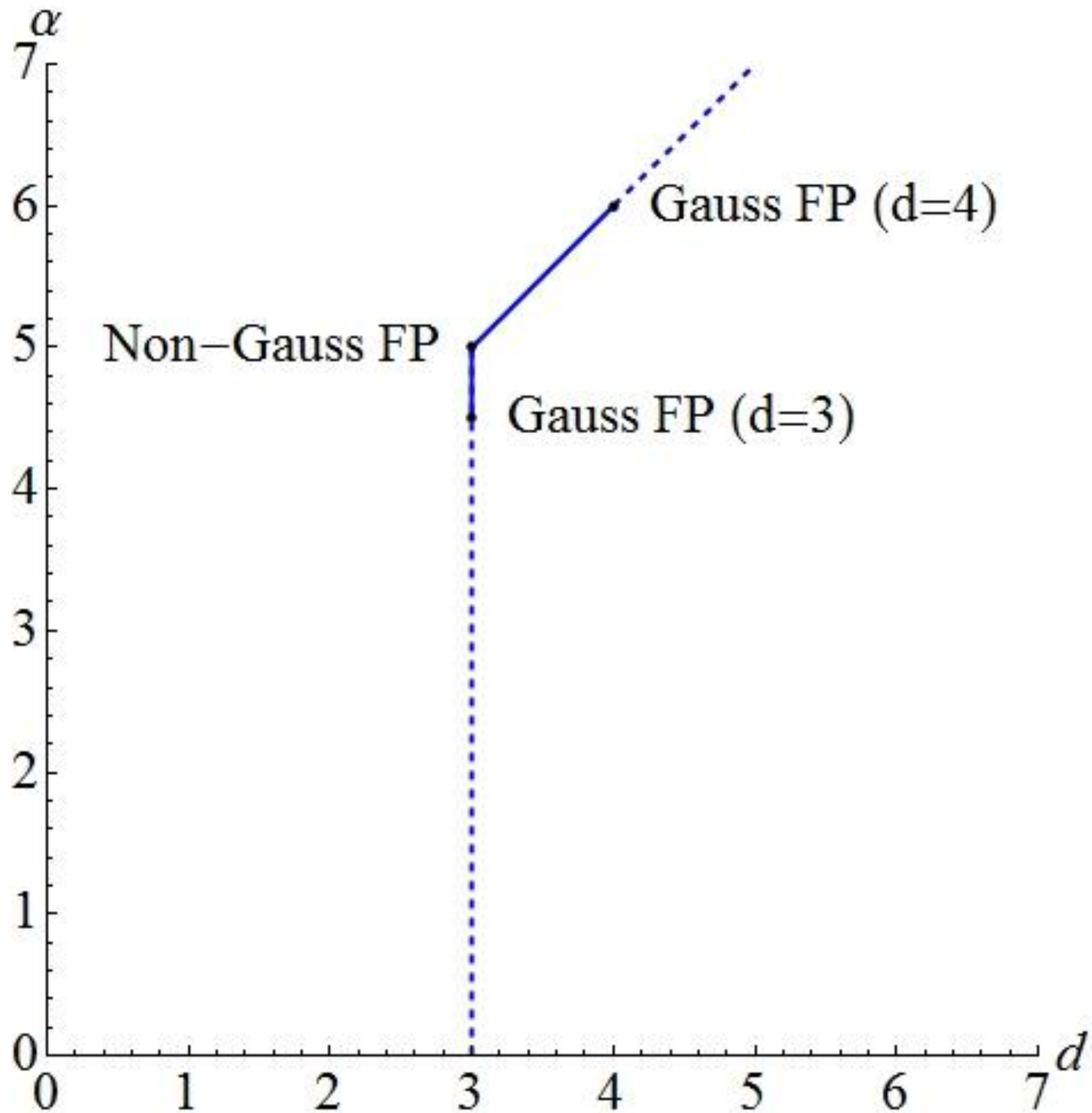
5. Special role of $\alpha = d$. It follows from the previous commutative relation and the fact that all fixed points and all cycles go to the singular point (-1,0) when α tends to d. What is analog of the singular point in the Euclidean models?

6.Similarity of $(\alpha - 3d/2)$-expansions for critical exponents in p-adic and Euclidean bosonic models .

7. If $\alpha \to d/2$, then "+"-fixed point tends to infinity (to the δ-function). Note, that δ-function fixed point corresponds to the "zero" automodel field. When $\alpha < d/2$ "+"-fixed point belongs to the upper half-plane again. In physical papers usually α is fixed and is equal to $(d + 2)$. In that case the Gaussian part of the Hamiltonian is given by the Laplace operator.

Physicists consider $(4-d)$ -expansion and try to extrapolate the results of the expansion to the point $d = 3$ (they have a few lower order members of the series with zero convergence radius). If we do the same in the (r, g)-space of the coupling constants of the fermionic hierarchical model, we will see that $d = 4$ is bifurcation value of the parameter d and we can construct $(4 - d)$-expansion from the Gaussian fixed point at the dimension $d = 4$.

From the explicit formulas for the "+"-fixed points it follows that $(\alpha - 3/2d)$-expansion and $(4 - d)$-expansion describe the same non-Gaussian fixed point at the dimension $d = 3$. We have some arguments that the same is true in the bosonic hierarchical model.How to verify physically interesting conjecture about the equivalence of $(\alpha - 3/2d)$-and $(4 - d)$-expansions in the Euclidean case?



α
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Gauss FP (d=4)
Non−Gauss FP
Gauss FP (d=3)
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d

References

1. Scalar models of p-adic quantum field theory and hierarchical model// Theor. Math. Phys., 1989,V.78,N2,P.248-257 (with Lerner E.Yu.)

2. P-adic Feynman and String amplitudes// Comm. Math. Phys., 1989, V.121, P.35-48, (with Lerner E.Yu.).

3. Renormalization group and renormalization theory in p-adic and adelic scalar models// Ed.Ya.G. Sinai: Dynamical Systems and Statistical Mechanics. Adv. in Sov. Math., V.3, Amer. Math. Soc. Providence, R.I., 1991, P.143-164.

4. Fixed points of renormalization group in the hierarchical fermionic model// J. Stat. Phys., 1994, V.76, P.805-817, (with Lerner E.Yu.).

5. Functional Integral via Functional Equation// Lett. Math. Phys., 1994, V.32, P.347-356.

6. Global flow of renormalization group and thermodynamic limit in the hierarchical fermionic model // Theor. Math. Phys., 1996, V.107, N.2, P.201-212.

7. Functional equations and renormalizations in p-adic field theories // Theor. Math. Phys., 1996, V.109, N.1, P.3-16

8. P-adic ϕ^4 -theory as a functional equation problem// Lett. Math. Phys., V.39, 253-260, 1997

9. RG-invariant curves in the fermionic hierarchical model Theor. Math. Phys., 1998, V.114, N.3, P.323-336

10. Critical phenomena in the fermionic hierarchical model. Theor. Math. Phys., 1998, V.117, N.3, P.471-488

11. Continuum limit in the fermionic hierarchical model// Theor. Math. Phys., 1999, V.118, N.1, P.40-50

12. Dynamical phenomena in the hierarchical fermionic model/ Phys. Lett., 1999, V.A253, P.41-46

13. Renormalization group solution of fermionic Dyson model// Asymptotic Combinatorics with Application to Mathem. Physics, V.A.Malyshev and A.M.Vershik (eds.), 2002, Kluwer Academic Publishers p.151-166

14. Epsilon-expansions in the hierarchical Dyson model// Theor. Math. Phys., 2004, V.139, N 2,p.268-275 (with R.G. Stepanov)

15. Renormalization group symmetry in p-adic models// Proc. of the Steklov Inst.of Mathematics, 2004, V.245, p.172-181

16. Epsilon-expansions in N-component ϕ^4-model// Theor. Math. Phys., 2006, V.146, N 3,p.365-384 (with R.G. Stepanov)

17. Critical exponents in p-adic ϕ^4-model// Amer. Inst. Phys. Conf. Proceedings, 2006, V.826, p.129-139

18. Vertex parts of p-adic Feynman amplitudes//Proc. of the Steklov Inst.of Mathematics, 2009, V.265, (with R.G. Stepanov)

19. P-Adic renormalization group solutions and the Euclidean renormalization group conjectures // P-Adic Numbers, Ultrametric Analysis and Applications.-2012.- V. 4, N. 2.-P. 109-114, Pleiades Publishing Ltd.

20. Renormalization group in the fermionic hierarchical model in the projective coordinates // Theor. Math. Phys. , 2012, V. 173, N 3, p. 355-362

21. Functional Fourier transformation and renormalization group transformation in the bosonic field models // Theor. Math. Phys. , 2013, V. 174, N 2, p. 303-312

22. An Algorithm for Studying the Renormalization Group Dynamics in the Projective Space //Proc. of the Steklov Inst.of Mathematics, 2014, Vol. 285, pp. 211221. (with A.F. Shamsutdinov)